Phase Lengths in the Cylic Cellular Automaton

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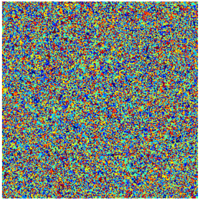
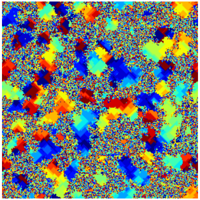
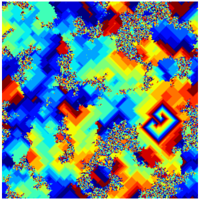
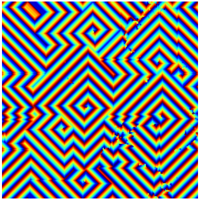
CSC 2019

Outline

1. Cyclic cellular automaton (CCA)
2. Phases in the CCA
3. Measuring phase lengths
4. Experimental design
5. Results

Spiral Wave Excitable Media

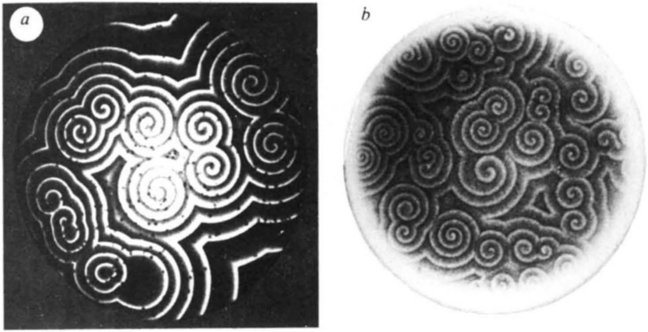
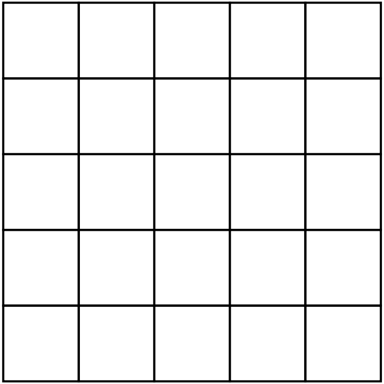


Figure from A. T. Winfree and S. H. Strogatz, "Organizing centres for three-dimensional chemical waves," *Nature*, vol. 311, pp. 611–615, 1984.

Cyclic Cellular Automaton (CCA)



(Fisch, Gravner, & Griffeath, 1991)

Cyclic Cellular Automaton (CCA)

$k = 9$

7	6	7	1	6
3	1	1	3	7
8	5	4	5	8
4	0	7	7	0
0	4	1	0	1

(Fisch, Gravner, & Griffeath, 1991)

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Cyclic Cellular Automaton (CCA)

$$\mathbf{t} \rightarrow t + 1$$

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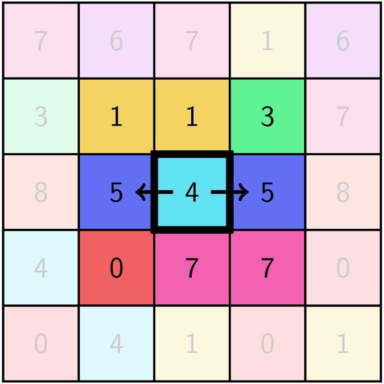

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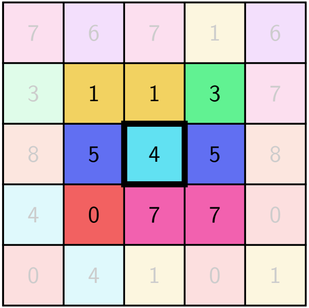
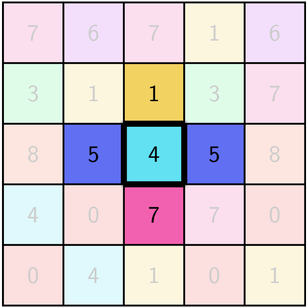
Cyclic Cellular Automaton (CCA)

Moore neighborhood



von Neumann neighborhood



Formal Definition

Notation

$\zeta_t(x) \in \{0, 1, \ldots, k-1\}$	state of cell x at time t
$\mathcal{N}(x)$	neighbors of cell x
$\mathcal{N}_t^+(x)$	promoters of cell x at time t

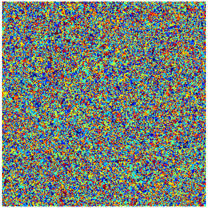
Update Rule

$$\mathcal{N}_t^+(x) = \{y \in \mathcal{N}(x) \mid (\zeta_t(x) + 1) \bmod k = \zeta_t(y)\}$$

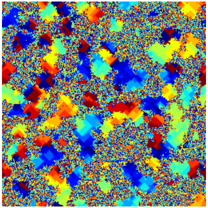
$$\zeta_{t+1}(x) = \begin{cases} (\zeta_t(x) + 1) \bmod k & \text{if } |\mathcal{N}_t^+(x)| \geq 1 \\ \zeta_t(x) & \text{otherwise} \end{cases}$$

CCA Phases

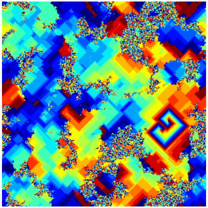
Debris



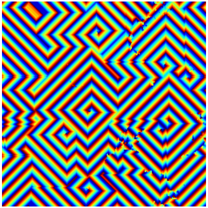
Droplet



Defect



Demon



(Fisch, Gravner, & Griffeath, 1991)

Prior CCA Work

1. Classifying behavior in parameter space
   (Fisch, Gravner, & Griffeath, 1991), (Durrett & Griffeath, 1993), (Hawick, 2013)
2. Effects of neighborhood on spiral shape
   (Reiter, 2010)
3. Quantifying self-organization
   (Shalizi & Shalizi, 2003)

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How does the number of cell types affect phase lengths?

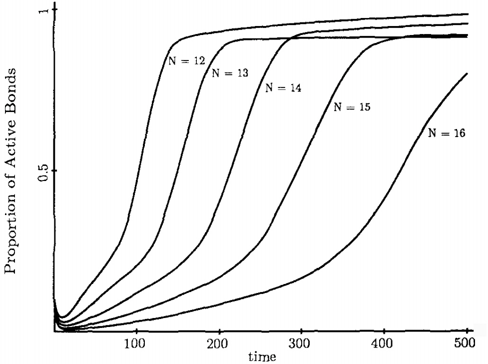
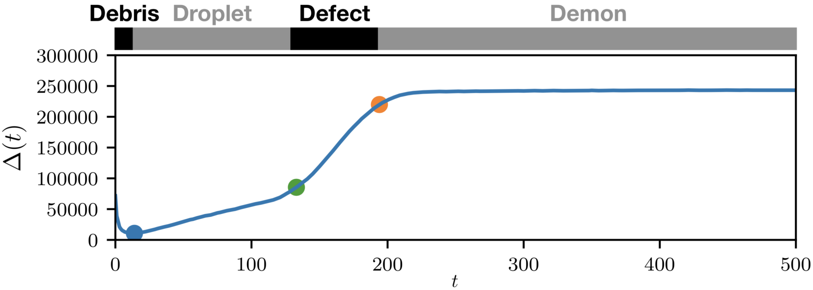


Figure from R. Fisch, J. Gravner, and D. Griffeath, "Cyclic cellular automata in two dimensions," in *Spatial Stochastic Processes*. Springer, 1991, pp. 71–185.

Identifying Phase Transitions

$$\Delta(t) = \sum_x (\zeta_t(x) - \zeta_{t-1}(x) \bmod k)$$



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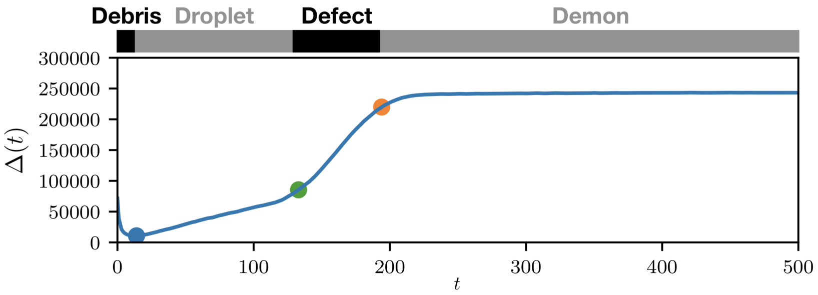
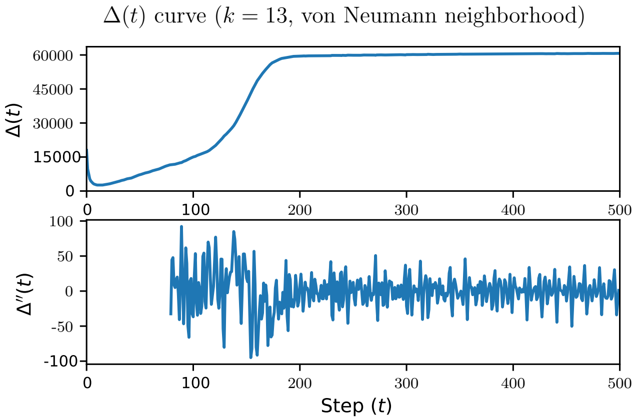


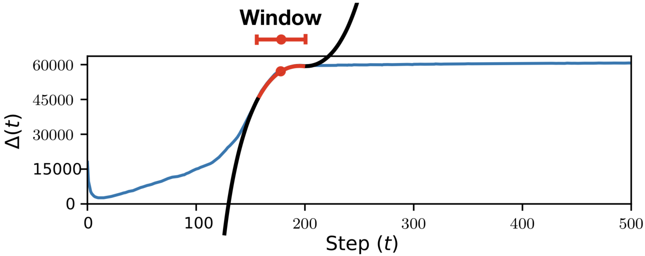
$\Longrightarrow$ transitions at local extrema of $\Delta''(t)$

Noise Amplification



Savitzky-Golay Differentiation

1. Pick window width around point
2. Fit low-degree polynomial to data in window
3. Take derivative of fitted polynomial at point



(Savitzky & Golay, 1964)

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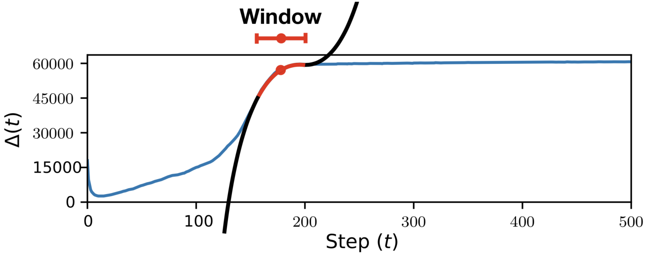

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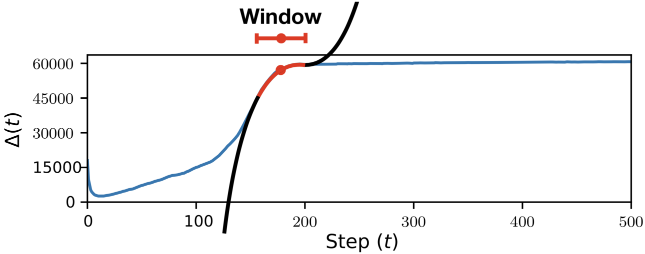

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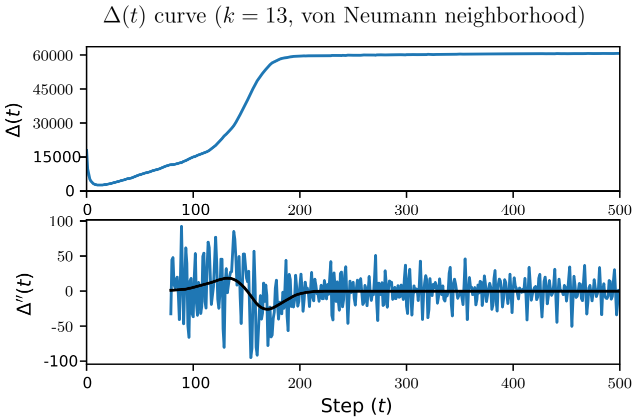
Identifying Phase Transitions



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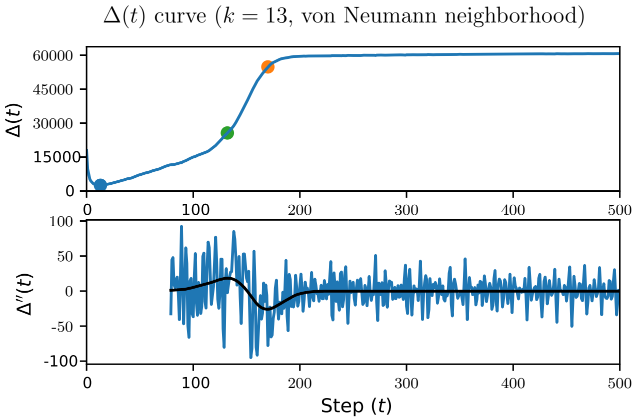

Simulation Procedure

- 1024 trials of 500 steps for each setting
- 3 grid sizes
- 2 neighborhoods
- k between 7 and 20

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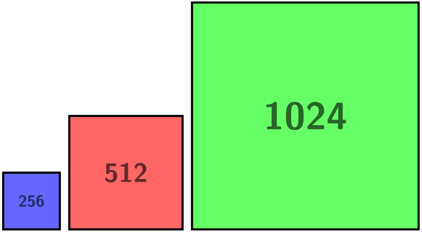

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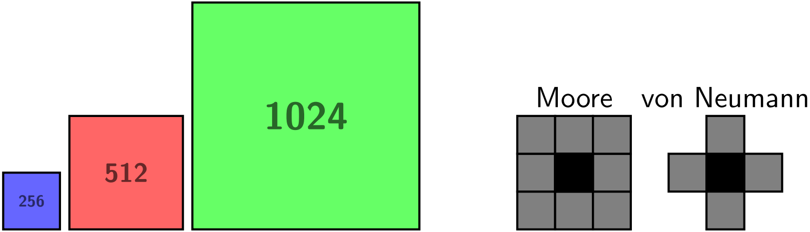

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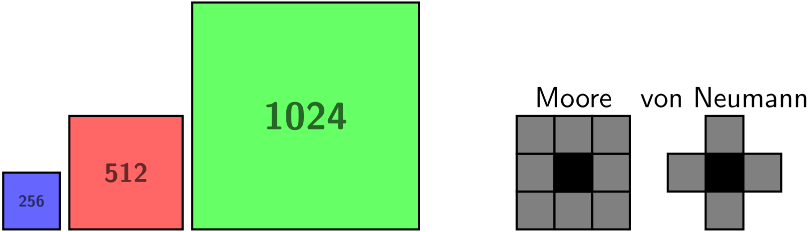

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Phase Length Dependence on k

For each parameter setting:

1. Compute phase lengths in each trial
2. Average over trials

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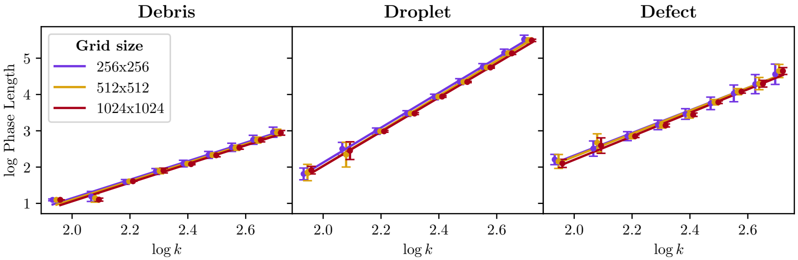

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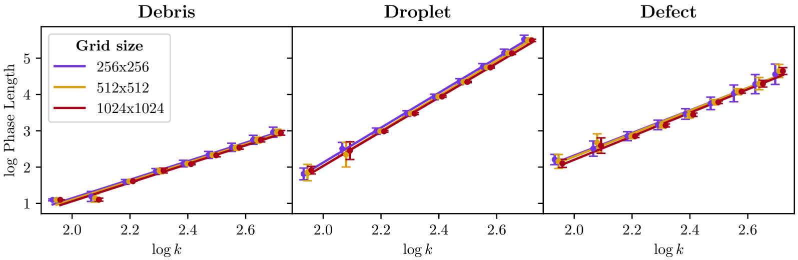

$$\log L = a + b \log k$$

$$L = ak^b$$

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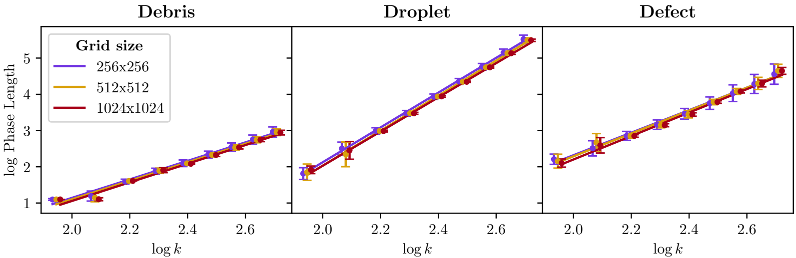

$$\log L = a + b \log k$$

$$L = \mathbf{a} k^{\mathbf{b}}$$

Power Law Exponents and Coefficients

Table 1: Phase length power law exponents

(a) von Neumann $\mathcal{N}$

	256×256	512×512	1024×1024	**Mean**
Debris	2.55 ± 0.01	2.55 ± 0.01	2.57 ± 0.00	2.56 ± 0.01
Droplet	4.81 ± 0.08	4.89 ± 0.22	4.75 ± 0.23	4.81 ± 0.07
Defect	3.08 ± 0.10	3.16 ± 0.11	3.24 ± 0.11	3.15 ± 0.06

(b) Moore $\mathcal{N}$

	256×256	512×512	1024×1024	**Mean**
Debris	2.52 ± 0.01	2.57 ± 0.01	2.60 ± 0.01	2.56 ± 0.01
Droplet	4.34 ± 0.06	4.32 ± 0.07	4.37 ± 0.07	4.34 ± 0.03
Defect	2.88 ± 0.13	2.82 ± 0.09	2.77 ± 0.10	2.81 ± 0.06

Table 2: Phase length power law log coefficients

(a) von Neumann $\mathcal{N}$

	256×256	512×512	1024×1024	**Mean**
Debris	-3.99 ± 0.02	-4.00 ± 0.02	-4.05 ± 0.01	-4.03 ± 0.01
Droplet	-7.58 ± 0.20	-7.80 ± 0.57	-7.42 ± 0.59	-7.59 ± 0.18
Defect	-3.88 ± 0.23	-4.05 ± 0.29	-4.25 ± 0.29	-4.03 ± 0.15

(b) Moore $\mathcal{N}$

	256×256	512×512	1024×1024	**Mean**
Debris	-5.02 ± 0.03	-5.14 ± 0.02	-5.24 ± 0.01	-5.20 ± 0.01
Droplet	-8.31 ± 0.16	-8.25 ± 0.16	-8.38 ± 0.20	-8.30 ± 0.10
Defect	-4.76 ± 0.34	-4.59 ± 0.25	-4.45 ± 0.26	-4.58 ± 0.16

Uncertainty Estimation

Random variance?

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$\Longrightarrow$ bootstrapping

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$\implies$ bootstrapping

Bias from Savitzky-Golay window widths?

Uncertainty Estimation

Random variance?

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Bias from Savitzky-Golay window widths?

$\Longrightarrow$ perturb widths

Uncertainty Estimation

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$\implies$ bootstrapping

Bias from Savitzky-Golay window widths?

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Conservative estimate: add confidence intervals

Different Phase Length Sensitivities

Moore $\mathcal{N}$

Phase	Exponent
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von Neumann $\mathcal{N}$

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- Smaller neighborhood $\implies$ more sensitive to k?

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Defect	3.15 ± 0.06

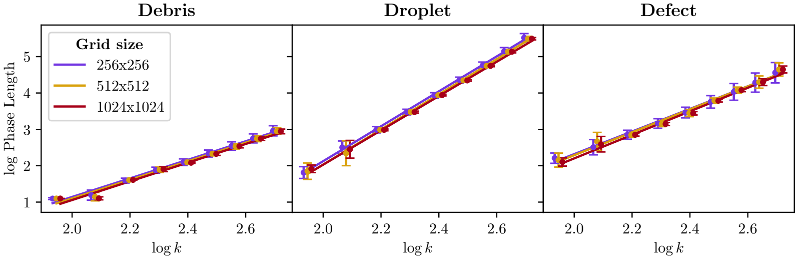
- Smaller neighborhood $\implies$ more sensitive to k?
- Droplet phase most sensitive

Other Methods for Finding Transitions

4	3	4		
3		5		
2		6	5	4
1				5
0	8	7	7	6

Identify defects directly?

Conclusions



1. CCA phase lengths depend on k via simple power laws
2. Independent of grid size
3. Exponent indicates sensitivity of phase to k

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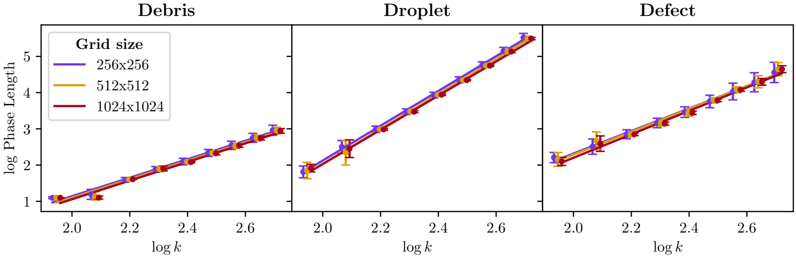

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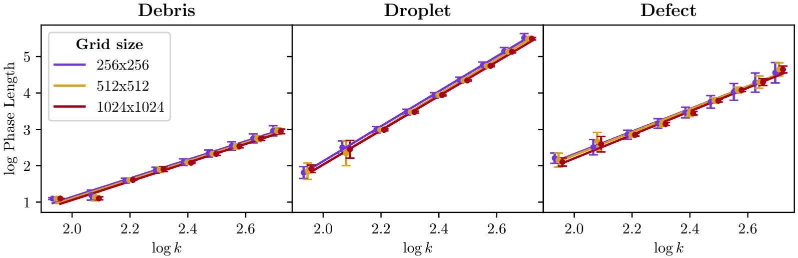

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Acknowledgments

- Travel and conference funding from Carleton College
- Thanks to Frank McNally for feedback